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A Theory of Smeared Continuum Damage Mechanics

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This paper considers smeared continuum damage mechanics based on the equivalent elliptical crack representation of a local damage. This approach provides a means of utilizing the crack energies derived in fracture mechanics, and of identifying the local damage state from local stress and strain information. The strain energy equivalence principle is used to derive the effective continuum elastic properties of a damaged solid in terms of the undamaged elastic properties and a scalar damage variable. The scalar damage variable is used to develop a consistent damage evolution equation. The combination of representing local damage as an equivalent elliptical crack, the determination of effective elastic properties using a strain energy equivalence principle, and a consistent damage evolution equation yields a simple, yet powerful local approach for continuum damage analysis

Key Words: Continuum Damage Mechanics, Elliptical Crack, Strain Energy Equivalence Principle(SEEP), Damage Evolution Equation, Effective Continuum Model

1. Introduction

In the half-century since the end of World War II, numerous approaches to accurately predict the remaining operational life of a mechanical system have appeared in the literature associated with various fields, including physics, applied mathematics, material sciences and engineering, fracture mechanics, and damage mechanics.

A material failure process is often assumed to involve a general degradation of elastic properties due to the highly localized nucleation and growth of microdefects (*i. e.*, microcracks and microvoids) and their ultimate coalescence into macrodefects. The process and result of these irreversible, energy dissipating, microstructural rearrangements is often called damage. Because of the complex nature of damage, there is no general agreement regarding the definition of damage variable(s). As Krajcinovic and Mastilovic (1995) discussed, selection of a damage variable is largely a matter of taste and convenience, and often has no obvious physical basis. Despite the non-uniqueness of damage definitions, much research has addressed the two major subjects of damage mechanics: the constitutive equations of damaged materials, and damage evolution laws. Extensive treatments of continuum damage mechanics can be found in the references by Krajcinovic (1989) and Lemaitre (1992). Currently existing damage mechanics theories for initially isotropic materials can be classified into four categories. The first category that permits isotropic behavior of damaged material, while using a scalar damage variable, includes the theories by Kachanov (1958), Rabotnov (1969), Leckie and Hayhurst (1977), Lemaitre (1985, 1986, 1992), Simo and Ju (1987), Fotiu *et al.* (1991), and others. The second category that permits isotropic behavior using tensor (or vector) variables includes Davison and Stevens (1973), Murakami and Ohno (1981), and others. The third category that permits anisotropic behavior using tensor (or vector) damage variables includes Krajcinovic and Fonseca (1981), Krajcinovic (1985, 1989), Lubarda *et al.* (1994), and others. To the author's knowledge, there have been no theories in the fourth category that permit

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anisotropic behavior of damaged material using a scalar damage variable. It is interesting to conclude in advance that the damage theory introduced in this paper falls under the fourth category. Very recently the continuum damage theory for the initially anisotropic materials has been developed by the author (Lee, 1997).

The constitutive equations of damaged material can be formulated using micromechanical and/or phenomenological approaches. The micromechanical modeling process leads to a one-to-one correspondence between a discontinuous field on an inhomogeneous mesoscale and an effective continuous field on the homogenous macroscale. The homogenization (averaging) of the mesostructural field of defects within a representative volume element (RVE) into a macrofield of the effective continuum corresponds to micromechanical modeling (*e. g.*, Sumarac and Krajcinovic(1987)). Despite clarity and a well-defined relationship with physical phenomena, it may be impractical or impossible to accurately realize the stochastic defects within a RVE, especially during the phases of crack generation and growth. In contrast to micromechanical models, phenomenological models do not consider the micro-details of material response, but describe damage indirectly by introducing internal (or hidden) variables. This has caused some confusion and spawned more extensive, substantially different, models of the same phenomena. Thus, to provide a scientific basis for theories of continuum damage mechanics, the irreversible thermodynamics has also been used (Murakami and Ohno, 1981; Krajcinovic, 1985; Lemaitre, 1985; Simo and Ju, 1987; Fotiu, *et al.*, 1991).

For damage evolution equations, the Kachanov equation (Kachanov, 1958) was the first of its kind. In the Kachanov equation, a scalar damage variable is defined as $0 \leq D \leq 1$ so that $D=0$ corresponds to the undamaged state, while $D=1$ is equivalent to complete local rupture of the material. Historically the Kachanov equation provided a basis for Rabotnov's effective stress concept (1969) and later for Lemaitre and Chaboche's effective stiffness concept (1985). Following the transition of damage interpretation, many researchers have focused on generalizing the one-dimensional constitutive equation of a damaged material to anisotropic damage states induced by a three-dimensional distribution of defects. Despite numerous developments, the loss of physical insight, the complexity of mathematical formulation, and the practical difficulties of measuring damage parameters restrict the applicability of many damage definitions available in the literature. Hence, the appropriate definition of damage variable(s) and the development of corresponding evolution equations and constitutive equations still seem to be open issues.

A Theory of Smeared Continuum Damage Mechanics In the literature, the constitutive equations and damage evolution equations are usually developed by using the concept of effective stress based on the strain equivalence principle (Lemaitre, 1985) or by using the concept of effective strain based on the stress equivalence principle (Simo and Ju, 1987). Although the effective stress or strain concepts can be converted to an effective stiffness concept via the strain or stress equivalence principles, and vice versa, choosing and consistently applying one of these principles may be confusing. This is probably why the strain equivalence principle is considered as just one possible principle in continuum damage mechanics (Lemaitre, 1992). The possibility of developing simpler more straightforward principles for use in continuum damage mechanics is a reasonable one to explore.

Thus, the purposes of this paper are: (1) to introduce a new theory of continuum damage mechanics based on the strain energy equivalence principle and the equivalent elliptical microcrack representation of a local damage and (2) to derive a consistent damage evolution equation from the damage variable definition and the crack growth law in fracture mechanics.

2. Features of the Present Continuum Damage Theory

It has been well-recognized that a material failure process involves a general degradation of effective elastic properties due to highly localized nucleation and growth of microdefects and their ultimate coalescence into microdefects. Any local damage associated with microdefects always locally changes the virgin isotropic properties into anisotropic properties. Thus, an observed change of elastic properties may indicate the existence of local damage. Based on this reasoning, it may be appropriate to assume that a local damage state can be equivalently represented by the effective anisotropic elastic properties. Then, in effect, a microdefect embedded in a solid is smeared smoothly onto an equivalent continuum having effective anisotropic elastic properties. By replacing all local damages in which microdefects exist with the effective anisotropic elastic properties, conventional stress analysis methods can be employed for damage or crack propagation analysis. This concept of the smeared continuum damage model seems to be well matched with the so-called local approach of fracture introduced by Lemaitre (1986). In his local approach, a crack edge is considered as a local zone in which damage increases until complete local failure of the material occurs. This may perhaps be considered a continuous version of crack propagation.

One of the key goals of this paper is to consider an efficient continuum local approach to represent a damaged material volume cell (MVC) containing a single microdefect into an equivalent (fictitious) continuum model (ECM) in terms of the effective continuum elastic properties and an appropriate damage variable. For this purpose, the equivalent continuum modeling (ECM) approach based on energy equivalence concepts (Lee, 1994) will be used. In the structural dynamics community, the ECM approach has proven capable of capturing the global behavior of discrete structures such as periodic lattice space structures. At the micro-level, damaged materials with microdefects may be regarded as discontinuous ones. Thus, the continuum representation of the mechanical behavior of such a material is likely to be in analogy with that of discrete structures. In the context of the ECM approach, *energy equivalence* means that the original structure and its equivalent continuum model must contain equal kinetic and strain energies when both are subject to the same global displacement and velocity fields.

In the development of an equivalent continuum model of a damaged material, it is assumed that the distances between small microdefects are sufficiently large so that each defect is affected only by the stress distribution around the microdefect. Then, a small material volume cell (MVC) that contains only a single defect can be isolated. The strain on the boundary of the MVC of the damaged material is taken to be same as that on the ECM. This assumption implies that the macro-behavior represented by the ECM is the same as that of the damaged material. From this observation, the strain energy equivalence principle (SEEP) may be introduced. This principle will be used to develop the effective continuum elastic properties of the damaged material as well as a new definition of damage variable.

Principle of Strain Energy Equivalence: When the MVC of the damaged material and its ECM volume cell have identical global displacements on their boundaries, they contain equal strain energy.

The SEEP may provide the effective continuum elastic properties of the ECM by equating the strain energy density v_d contained in the MVC of the damaged material to the strain energy density v_{eq} in the corresponding ECM. That is,

$$v_{eq}(\bar{C}\ ;\ \varepsilon) = v_d(C, D\ ;\ \varepsilon) \qquad (1)$$

where C represents the elastic stiffness of the undamaged material, $\bar{C}$ the effective continuum elastic stiffness, D the damage variable consistent with $\bar{C}$, and ε represents the average strain on the boundaries of the MVC and ECM. Thus, complete constitutive equations for coupled elasticity and damage may be obtained by replacing the undamaged elastic stiffness with the effective continuum elastic stiffness $\bar{C}$, calculated from Eq. (1), without redefining or changing the nominal stress or strain appearing in the original constitutive equations. This seems consistent with the physical interpretation of elastic modulus degradation due to reduction in the effective stress-transmitting area (Lubarda, *et al.* 1994).

The strain energy density v_d in Eq. (1) is

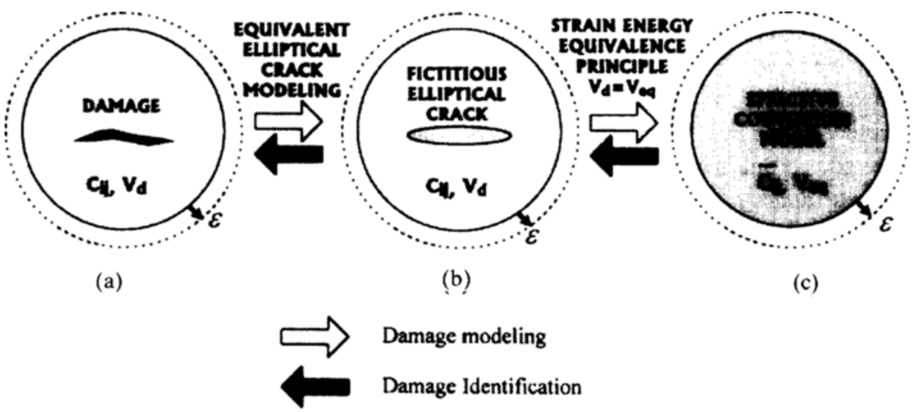


Fig. 1 General features of smeared continuum damage modeling approach: (a) current damaged state of a material volume cell (MVC) subject to boundary strain ε; (b) equivalent elliptical crack representation of a local damage; and (c) effective continuum model (ECM) representation (subject to the same boundary strain ε) based on strain energy equivalence principle (SEEP).

dependent of the damage variable D, which characterizes the current damage state, *i. e.*, its geometry and growth direction. Unfortunately, for internal microdefects, the current damage state is inaccessible. Thus, in practice, it is perhaps impossible to identify the damage state in detail. Hence, prediction of damage evolution and fracture using the continuum approach based on SEEP requires a method by which the damage state can be determined at some time. In addition, there should be a relation with which current damage information can be converted to an effective continuum representation. Since it is common for any microdefect (or damage in general sense) to reduce the effective elastic properties, various kinds of microdefects may be considered as the equivalent (fictitious) elliptical microcracks which result in the same degradation of elastic properties. In other words, an elliptical microcrack may be considered as a construct which relates some general damage to the effective elastic properties of an ECM. This equivalent elliptical microcrack modeling of local damage may bring several benefits. Firstly, for certain elliptical microcracks embedded in two- and three-dimensional elastic solids, the closed-form solutions are available from fracture mechanics, and they can be used for deriving the strain energy density v_d and thus for determining effective elastic properties from Eq. (1). Secondly, the damage state (geometry and growth direction) may be characterized by determining the aspect ratio and crack coordinate directions of an equivalent elliptical microcrack model. The combination of representing local damage as an equivalent elliptical microcrack and the determination of effective elastic properties using SEEP completes the present smeared continuum damage modeling. Figure 1 illustrates the general features of the present smeared continuum damage theory. In the following two sections, the determination of the effective continuum elastic stiffness based on SEEP and the determination of elliptical microcrack characteristics will be addressed.

3. Effective Continuum Model and a Scalar Damage Variable

To develop a damage model based on SEEP, the change of the strain energy storage capacity of an elastic body due to the presence of elliptical microdefects is considered. A damaged body cannot store as much strain energy under a given deformation as an undamaged body because of the degradation of elastic moduli. In fracture mechanics, the change in strain energy associated with forming new surfaces in a body has been explored over many decades. The strain energy released in forming a crack is often called the crack energy. Crack energies forelliptical through-cracks in an infinite two-dimensional isotropic elastic body and for elliptical surface cracks embedded in an infinite three-dimensional isotropic elastic body have been calculated by Sih and Liebowitz (1967). In order to apply their results to a MVC with a microcrack, the microcrack size is assumed to be relatively small compared to the characteristic dimension of a MVC, so that the crack energy in a MVC may be approximated by that of an infinite body. This approximation is appropriate in that (1) the effects of neighboring cracks decay rapidly with distance (Rice, 1968); (2) complete local failure of material is likely to occur far before the damage variable D reaches the value of 1 (Lemaitre, 1985).

In the present work, the undamaged material is assumed to behave isotropically. Nucleation and growth of damage, however, may cause the equivalent damaged material to behave anisotropically. The stress-strain relation for the ECM of a damaged material may be written in terms of the damaged (or effective continuum) elastic stiffness $\bar{C}_{ij}$, as

$$\{\sigma\}=[\bar{C}_{ij}]\{\varepsilon\} \tag{2}$$

The effective continuum elastic stiffness $\bar{C}_{ij}$ is determined from the undamaged isotropic elastic stiffness C_{ij} and the new damage variable D.

The strain energy density v_{eq} stored in the ECM can be written in the form

$$v_{eq}=\frac{1}{2}\{\varepsilon\}^T[\bar{C}_{ij}]\{\varepsilon\} \tag{3}$$

The strain energy density v_d that can be stored in a MVC of damaged material is obtained by subtracting the strain energy v_c released by an elliptical crack (*i. e.*, crack energy) from the strain energy density v_0 in undamaged isotropic elastic body; that is

$$v_d=v_0-v_c \tag{4}$$

In this paper, two different elliptical cracks in isotropic elastic bodies are considered. They are (1) an elliptical through crack in an two-dimensional body (simply 2-D crack), and (2) an elliptical plane crack embedded in an infinite three-dimensional body (or more simply a 3-D crack). The origin of a rectangular Cartesian coordinate system (crack coordinates) is located at the centers of 2-D and 3-D elliptical cracks, and the coordinate directions 1 and 2 are aligned with the major axis (length $2a$) and the minor axis (length $2b$) of the crack, respectively. For the 3-D crack, the coordinate 3 is normal to the plane of the crack.

From Sih and Liebowitz (1967), the average strain energy density v_d which can be stored in a MVC (characteristic region of radius R) of initially isotropic elastic solid containing an elliptical crack can be readily derived. The strain energy densities v_d originally derived by Sih and Liebowitz were in terms of the applied stresses at infinite distance from cracks. Since the applied stresses at infinity can be replaced with the equivalent strains at infinity, using the stress-strain relations of the *undamaged* solid, their results are rewritten in terms of the equivalent strains at infinity.

For a two-dimensional solid containing an elliptical through-crack (*i. e.*, 2-D crack), the strain emegy density is

$$v_d=\frac{1}{2}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix}^T\cdot\begin{bmatrix}C_{11}(1-e_{11}D) & C_{12}(1-e_{12}D) & 0\\ C_{12}(1-e_{12}D) & C_{22}(1-e_{22}D) & 0\\ 0 & 0 & C_{66}(1-e_{66}D)\end{bmatrix}\cdot\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{Bmatrix} \tag{5}$$

where $D=(a/R)^2$, and the parameters e_{ij} are defined, for plane stress, as

$$\begin{aligned}e_{11}&=\left(\frac{2v^2}{1-v^2}\right)+\left(\frac{1-v}{1+v}\right)k'+\left(\frac{2}{1-v^2}\right)k'^2\\ e_{22}&=\left(\frac{2}{1-v^2}\right)+\left(\frac{1-v}{1+v}\right)k'+\left(\frac{2v^2}{1-v^2}\right)k'^2\\ e_{12}&=\left(\frac{2}{1-v^2}\right)-\left(\frac{1-v}{v(1+v)}\right)k'+\left(\frac{2}{1-v^2}\right)k'^2\\ e_{66}&=\left(\frac{1}{1+v}\right)(1+k')^2\end{aligned} \tag{6}$$

while, for plane strain, as

$$e_{11}=\left(\frac{2v^2}{1-2v}\right)+(1-2v)k'+\left(\frac{2(1-v)^2}{1-2v}\right)k'^2$$
$$e_{22}=\left(\frac{2(1-v)^2}{1-2v}\right)+(1-2v)k'+\left(\frac{2v^2}{1-2v}\right)k'^2$$
$$e_{12}=\left(\frac{2(1-v)^2}{1-2v}\right)-\left(\frac{(1-v)(1-2v)}{v}\right)k'+\left(\frac{2(1-v)^2}{1-2v}\right)k'^2$$
$$e_{66}=(1-v)(1+k')^2 \tag{7}$$

In the preceding equations, $k'=b/a$ represents the aspect ratio of an elliptical crack with values $0\leq k'\leq 1$ (same for the 3-D crack case). From the definition, $k'=0$ represents a line crack while $k'=1$ represents a circular crack. The C_{ij} are the isotropic elastic stiffness coefficients of the undamaged solid. They are defined in terms of the usual engineering constants (Young's modulus E, shear modulus G, and Poisson's ratio v), for plane stress, as

$$C_{11}=C_{22}=\frac{E}{1-v^2}$$
$$C_{12}=\frac{vE}{1-v^2}$$
$$C_{66}=G \tag{8}$$

and, for plane-strain case, as

$$C_{11}=C_{22}=\frac{(1-v)E}{(1+v)(1-2v)}$$
$$C_{12}=\frac{vE}{(1+v)(1-2v)}$$
$$C_{66}=G \tag{9}$$

Similarly, for a three-dimensional solid containing an elliptical plane crack (*i. e.*, 3-D crack), the strain energy density is

$$v_d=\frac{1}{2}\left(\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{Bmatrix}^T\cdot\begin{bmatrix}C_{11}(1-e_{11}D) & C_{12}(1-e_{12}D) & C_{12}(1-e_{13}D)\\ C_{12}(1-e_{12}D) & C_{11}(1-e_{11}D) & C_{12}(1-e_{13}D)\\ C_{12}(1-e_{13}D) & C_{12}(1-e_{13}D) & C_{11}(1-e_{33}D)\end{bmatrix}\cdot\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{Bmatrix}+\begin{Bmatrix}\gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{Bmatrix}^T\begin{bmatrix}C_{66}(1-e_{44}D) & 0 & 0\\ 0 & C_{66}(1-e_{55}D) & 0\\ 0 & 0 & C_{66}\end{bmatrix}\begin{Bmatrix}\gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{Bmatrix}\right) \tag{10}$$

where $D=(a/R)^3$, and the parameters e_{ij} are defined as

$$e_{11}=e_{22}=\left(\frac{2v^2}{1-2v}\right)H(k)$$
$$e_{33}=\left(\frac{2(1-v)^2}{1-2v}\right)H(k)$$
$$e_{12}=\left(\frac{2v(1-v)}{1-2v}\right)H(k)$$
$$e_{13}=e_{23}=e_{33}$$
$$e_{44}=\frac{(1-v)k^2H(k)}{(k^2+vk'^2)-vH(k)K(k)}$$
$$e_{55}=\frac{(1-v)k^2H(k)}{(k^2-v)+vH(k)K(k)}$$
$$H(k)=\frac{k'^2}{E(k)} \qquad k^2=1-k'^2 \tag{11}$$

where, $E(k)$ and $K(k)$ are the complete elliptic integrals of the first and the second kinds, respectively. In Eq. (10), the C_{ij} are given by Eq. (9). Note that e_{55} is always larger than e_{44} for any Poisson ratio $v\leq 0.5$ and aspect ratio $0\leq k'\leq 1$, and they have the same maximum value of $4(1-v)/\pi(2-v)$ for circular plane cracks (*i.e.*, $k'=1$).

From Eqs. (1), (3), (5), and (10), the effective continuum elastic stiffness $\bar{C}_{ij}$ of ECM, for both 2-D and 3-D cracks, are found in simple form as

$$\bar{C}_{ij}=C_{ij}(1-e_{ij}D) \tag{12}$$

In detail, they are given, for the 2-D crack, as

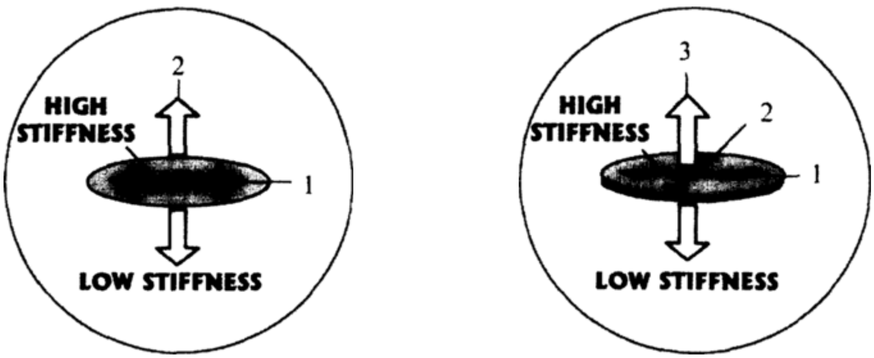
$$\bar{C}_{11}=C_{11}(1-e_{11}D) \quad \bar{C}_{12}=C_{12}(1-e_{12}D)$$
$$\bar{C}_{22}=C_{22}(1-e_{22}D) \quad \bar{C}_{66}=C_{66}(1-e_{66}D) \tag{13}$$

and, for 3-D crack, as

$$\bar{C}_{11}=C_{11}(1-e_{11}D) \quad \bar{C}_{33}=C_{11}(1-e_{33}D)$$
$$\bar{C}_{12}=C_{12}(1-e_{12}D) \quad \bar{C}_{13}=C_{12}(1-e_{13}D)$$
$$\bar{C}_{44}=C_{66}(1-e_{44}D) \quad \bar{C}_{55}=C_{66}(1-e_{55}D)$$
$$\bar{C}_{66}=C_{66} \tag{14}$$

The parameter D, defined as $(a/R)^2$ for the 2-D crack and $(a/R)^3$ for 3-D crack, will be considered as a scalar damage variable throughout this paper. This new damage variable D may be interpreted as the ratio of the effective damaged area (or volume) to the total area (or volume) of the MVC, which somewhat differs from the damage variable D used in classical theories of damage mechanics (*e. g.*, Lemaitre, 1992).

Equation (13), for the 2-D crack, shows that $\bar{C}_{11}=\bar{C}_{22}$ and $\bar{C}_{66}=(\bar{C}_{11}-\bar{C}_{12})/2$ when $k'=1$ (*i. e.*, circular crack). Thus, a damaged local area of



(a) Damaged with an elliptical throught -crack in 2-D solid (b) Damaged with an elliptical plane-crack in 3-D solid

Fig. 2 Directions of the highest and lowest effective continuum elastic stiffnesses of effective continuum model (ECM).

two-dimensional solid with a circular through-crack retains isotropic behaviors. When k' is not equal to 1, however, the damaged local area behaves orthotropically. In this case, the effective elastic stiffness $\bar{C}_{22}$ is always smaller than $\bar{C}_{11}$. This is consistent with results from fracture mechanics, namely that the presence of the highest stress intensity at the crack edge along the major axis effectively reduces the elastic stiffness in the minor axis direction, as illustrated in Fig. 2(a). In addition, the effective reduction in stiffness associated with the crack may well encourage crack propagation in a direction nearly aligned with the major axis.

For 3-D cracks, Eq. (14) also shows that a damaged local volume of three-dimensional solid with an embedded elliptical plane crack behaves orthotropically. Also observe that, since e_{33} is always larger than e_{11} or e_{22}, the effective continuum elastic stiffness $\bar{C}_{33}$ in the direction normal to the crack surface is the softest, as illustrated in Fig. 2(b). For a penny-shaped (circular) crack, $\bar{C}_{44}$ becomes identical to $\bar{C}_{55}$ since $e_{44}=e_{55}=4(1-v)/\pi(2-v)$, resulting in transversely isotropic behavior.

4. Determination of Current Local Damage State

As can be seen from the effective continuum elastic stiffness $\bar{C}_{ij}$ considered in the preceding section, local damage associated with an elliptical (non-circular) microcrack within a MVC always changes the virgin isotropic properties into orthotropic properties. Thus, an observed change from isotropic to orthotropic behavior seems to imply the existence of local damage. Since local damage is approximately represented as an equivalent (fictitious) elliptical microcrack, to determine the current state of an equivalent elliptical microcrack is identical to identifying the local damage state. However, unfortunately internal damages are inaccessible. Hence, a method by which the current state of elliptical microcrack model can be determined is required.

Since the effective continuum elastic stiffness $\bar{C}_{ij}$ was developed with respect to the crack coordinates, the orientation of the crack relative to the global structural coordinates must be determined. Once we know the local crack coordinate orientations and aspect ratio, the effective continuum elastic stiffness $\bar{C}_{ij}$ can be determined from Eqs. (13) or (14), which can then be transformed back to the global structural coordinates and used in the next step of an incremental calculation process. Thus, the determination of crack coordinates and aspect ratio of an equivalent elliptical microcrack is essential for the prediction of damage propagation. As stress analysis is typically conducted in the course of an incremental damage analysis, the current values of stresses and strains at a damaged local point are assumed to be available. Determination of the crack coordinate directions and the crack aspect ratio from this information will be considered in this paper.

The benefit from this analysis comes from the assumption of elliptical microcracks, in that the associated damage results in locally orthotropic material behavior. The orthotropic material principal coordinates may thus be considered to be aligned with the crack coordinates. The principal stress or principal strain directions are possible choices for this orientation because of their biaxial nature. When the opening mode crack growth is dominant, the material principal directions (crack coordinates) may be considered as the principal stress directions or the orientation that minimizes strain energy associated with shear deformation, in the sense of the first order approximation. As shown in Fig. 2, the likely direction of damage propagation is somewhat normal to the high stiffness direction, i. e., the "1" direction for 2-D crack and the "3" direction for 3-D.

4.1 Two-dimensional damage

As discussed in the preceding section, alternate approaches may be used to determine the principal material directions of the orthotropic damaged material. Assuming that the crack orientation has been established to within a rotation of 90 degrees using one of these approaches, the "1" and "2" directions remain to be determined. Given the stresses and strains referred to the global structural coordinate system, the stresses and strains in the crack coordinate system (I , II) may be readily found. Furthermore, the material moduli (inverse compliances, $1/S_{\mathrm{I\,I}}$ and $1/S_{\mathrm{II\,II}}$) associated with uniaxial stresses in each of the orthogonal directions which define the crack coordinate system may be determined. If these are identical, the material is locally isotropic, either having no local damage or circular microcracks. Otherwise, the direction associated with the highest stiffness corresponds to the "1" direction, the major axis of the elliptical crack.

From Eq. (13), the shear stress-strain relation may be used to establish the current aspect ratio k'. Assuming that the stresses and strains in the crack coordinate system approximate the principal stresses and strains, the effective maximum shear stress and strain are estimated as

$$\tau=\frac{|\sigma_{\mathrm{I}}-\sigma_{\mathrm{II}}|}{2} \qquad \gamma=\frac{|\varepsilon_{\mathrm{I}}-\varepsilon_{\mathrm{II}}|}{2} \tag{15}$$

These yield an estimate for the effective elastic shear stiffness

$$G^*=\frac{\tau}{\gamma} \tag{16}$$

Assuming the computed value of G^* is identical to the effective continuum shear stiffness of Eq. (13), the aspect ratio k' may be computed using

$$k'=1-\left[\frac{1}{aD}\left(1-\frac{G^*}{G}\right)\right]^{\frac{1}{2}} \tag{17}$$

where $a=1/(1+v)$ for plane stress and $a=(1-v)$ for plane strain. In general, $G^*/G\leq 1$ since G^* is softened from the virgin value of G.

4.2 Three-dimensional damage

A similar approach is used to address three-dimensional damage. Once the crack coordinate directions with respect to global coordinates are established by an alternate approach, then stresses and strains in that coordinate system may be calculated and, by considering material moduli under uniaxial stress, the "1," "2," and "3" directions may be distinguished. The "softest" direction under uniaxial normal stress corresponds to the normal to the planar crack, the "3" direction. The remaining problem is determining how to match the two principal material directions I and II to the crack coordinates 1 and 2.From Eq. (14), the effective elastic shear stiffnesses $\overline{C}_{44}$ and $\overline{C}_{55}$ can be used to distinguish the major axis of the elliptical surface crack from the minor axis; $\overline{C}_{11}$ and $\overline{C}_{22}$ are not useful for this purpose since they are identical. From Eq. (14), however, $\overline{C}_{55}$ is clearly softer than $\overline{C}_{44}$ for any $0\leq k'\leq 1$, and both are always softer than $\overline{C}_{66}=G$, the stiffness of the virgin material. Thus the "stiffest" direction under uniaxial shear stress corresponds to the "1" direction.

From the stresses and strains in the principal material coordinate system, the approximate effective maximum or minimum shear stresses and strains may be estimated as:

$$\tau_{\mathrm{II,III}}=\frac{|\sigma_{\mathrm{II}}-\sigma_{\mathrm{III}}|}{2} \qquad \gamma_{\mathrm{II,III}}=\frac{|\varepsilon_{\mathrm{II}}-\varepsilon_{\mathrm{III}}|}{2}$$
$$\tau_{\mathrm{I,III}}=\frac{|\sigma_{\mathrm{I}}-\sigma_{\mathrm{III}}|}{2} \qquad \gamma_{\mathrm{I,III}}=\frac{|\varepsilon_{\mathrm{I}}-\varepsilon_{\mathrm{III}}|}{2} \tag{18}$$

The current local effective elastic shear stiffnesses are now estimated from

$$G_{44}^{*}=\frac{\tau_{\mathrm{II,III}}}{\gamma_{\mathrm{II,III}}} \qquad G_{55}^{*}=\frac{\tau_{\mathrm{I,III}}}{\gamma_{\mathrm{I,III}}} \tag{19}$$

By comparing these estimated effective shear stiffnesses, the major axis of the equivalent elliptical crack, "1", is seen to correspond to the principal stress direction I if $G_{55}^{*}<G_{44}^{*}$, and vice versa.

To predict the current aspect ratio k', the estimated values of G_{44}^{*} and G_{55}^{*} are assumed to be identical to $\bar{C}_{44}$ and $\bar{C}_{55}$, respectively. Since e_{55} is always larger than e_{44} for any Poisson ratio $v\leq 0.5$ and aspect ratio $0\leq k'\leq 1$, $G_{55}^{*}<G_{44}^{*}$ is confirmed. Note that both e_{44} and e_{55} have the same maximum value of $4(1-v)/\pi(2-v)$ when $k'=1$, which does not exceed one. Finally, the aspect ratio k' for a given current damage D may be readily estimated using

$$e_{55}(v,k')-e_{44}(v,k')\frac{G_{55}^{*}}{G_{44}^{*}}=\frac{1}{D}\left(1-\frac{G_{55}^{*}}{G_{44}^{*}}\right) \tag{20}$$

5. Consistent Damage Evolution Equation

In the present paper, the scalar damage variable D is defined as

$$D=\left(\frac{a}{R}\right)^{n} \tag{21}$$

where $n=2$ for 2-D cracks and $n=3$ for 3-D cracks. Herein the definition of Eq. (21) will be used to develop a consistent damage evolution equation. Differentiation of Eq. (21) with respect to time and eliminating the half size of the crack, a, yields

$$\dot{D}\propto D^{\left(1-\frac{1}{n}\right)}\dot{a} \tag{22}$$

Since the microcrack size is not available in general, the crack growth rate,, should be expressed in terms of measurable or predictable quantities. Fracture mechanics has shown that slow crack growth is related to the stress intensity factor according to Paris's law (Paris and Erdogan, 1963). That is

$$\dot{a}=AK_I^N \qquad K_I=Y\sigma_n\sqrt{a} \tag{23}$$

where A and N are parameters to be determined from experiments, and K_I is the stress intensity factor, which depends on crack length, applied stress and a geometrical factor Y. Combining the relations (23) through (24) yields

$$\dot{D}\propto D^{\left(1-\frac{1}{n}+\frac{N}{2n}\right)}\sigma_n^N \tag{24}$$

In the sense of Paris's law, the stress σ_n in Eq. (24) should be the stress applied normal to the mid-plane between two microcrack surfaces. Therefore $\sigma_n=\sigma_2$ for two-dimensional damage, and $\sigma_n=\sigma_3$ for three-dimensional damage. A damage threshold may be introduced using the Heaviside step function $H(\sigma_{eq}-\sigma_{TH})$, as

$$\dot{D}=\beta D^{\left(1-\frac{1}{n}+\frac{N}{2n}\right)}\sigma_n^N H(\sigma_{eq}-\sigma_{TH}) \tag{25}$$

where β is a material constant to be determined from experiments, and σ_{TH} is a threshold stress above which damage will grow. To accommodate general three dimensional stress states, the driving stress in Eq. (25) has been replaced by σ_{eq}, the von Mises equivalent stress calculated from the deviatoric stresses (Lemaitre, 1992). Alternatively, the so-called damage equivalence stress may be used in place of the von Mises equivalent stress (Lemaitre, 1992), defined as

$$\bar{\sigma}=\sigma_{eq}\left(\frac{2}{3}(1+v)+3(1-2v)\left(\frac{\sigma_H}{\sigma_{eq}}\right)^2\right)^{\frac{1}{2}} \tag{26}$$

where σ_H denotes the hydrostatic stress. In order to determine the parameters β and σ_{TH} in Eq. (25), an experimental approach similar to that of Lemaitre (1992) may be followed. It should be pointed out that neither the von Mises stress nor the damage equivalence stress may be ideal for use in this application, as both were developed for use with isotropic materials. An alternative for future research might involve a new damage growth criterion.

6. Conclusions

To accommodate damage associated with general microdefects and to model the progress of local fracture in a continuum sense, the concept of an equivalent elliptical microcrack representation of local damage was introduced and explored. A strain energy equivalence principle was developed to derive the effective continuum elastic stiffnesses of two-dimensional and three-dimensional cracked solids. A scalar damage variable with a physical interpretation as a crack area or volume fraction was used to develop a consistent damage evolution equation. These features may be combined in an iterative incremental stress analysis to provide (Fang *et al*. 1997) to provide a basis for a continuum approach to crack propagation analysis.

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